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AHMADU BELLO UNIVERSITY, ZARIA

FIRST SEMESTER EXAMINATION 2023/2024
MATH 307: COMPLEX ANALYSIS I

DATE: April, 2024 **TIME: 2 HOURS**

INSTRUCTIONS: ATTEMPT ANY FOUR QUESTIONS

1. If z is a complex variable, prove the following identities:
   (a) $\sec^2 z = 1 + \tan^2 z$ (b) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
   (c) $\cosh(z_1 - z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2$ (d) $\coth^{-1} z = \dfrac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right)$

2. (a) Evaluate the following limits (i) $\lim\limits_{z\to 2i} \dfrac{(z+3)(z-7)}{2z+i}$ (ii) $\lim\limits_{z\to 4i} \dfrac{z^2+25}{z-5i}$ (iii) $\lim\limits_{n\to\infty} \dfrac{2n-3i}{2n+i}$
   (b) Prove that a single valued complex function $f(z)$ is continuous at a point z_0 if and only if $\lim\limits_{z\to z_0} f(z) = f(z_0)$
   (c) Investigate the continuity of $f(z) = \begin{cases} \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z-i} & \text{for } z \neq i \\ 4+4i & \text{for } z = i \end{cases}$ at $z = i$.

3. (a) Prove that if *f(z)* is continuous in a region, then the real and imaginary parts of *f(z)* are also continuous in the region.
   (b) (i) Prove that $\lim\limits_{n\to\infty} \dfrac{2n-5i}{2n+3i} = 1$ (ii) investigate the convergence of $\sum\limits_{n=0}^{\infty} \dfrac{(2-i)^n}{(z-i)^{n+1}}$
   (iii) Distinguish between absolute and conditional convergence of series.

4. (a) Prove that if $\sum\limits_{n=1}^{\infty} z_n$ converges to A and $\sum\limits_{n=1}^{\infty} w_n$ converges to B, then $\sum\limits_{n=1}^{\infty} (z_n - w_n)$ converges to A-B
   (b) Prove that if a series of complex numbers $\sum\limits_{n=1}^{\infty} z_n$ converges, then necessarily, $\lim\limits_{n\to\infty} z_n = 0$.
   (c) (i) Prove that $\lim\limits_{n\to\infty} z^n = 0$ *if* $|z| < 1$ (ii) Evaluate $\lim\limits_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)$

5. (a) Derive Cauchy Riemann equations in Cartesian form.
   (b) Investigate the differentiability of the following complex functions
   (i) $f(z) = x^2 + iy^2$ (ii) $f(z) = y^2 - ix^2$

6. (a) State and prove the Cauchy's Integral Formula
   (b) Use the formula or otherwise to evaluate the following integral
   $\oint_C \dfrac{ze^{2z}+1}{(z-3)(z^2-1)}dz$ where C: rectangle $-2 \le x \le 2$; $\dfrac{-3}{2} \le y \le \dfrac{3}{2}$

$\frac{25}{5i} = \frac{5}{i}$

$\frac{-8}{5} + 5i$ $\frac{-8+25i}{5}$ $\frac{5i}{-1}$

$-8i - 25$

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(1a) $\sec^2 z = 1 + \tan^2 z$, Recall that $\sec z = \frac{2}{e^{iz}+e^{-iz}} \Rightarrow \sec^2 z = \left(\frac{2}{e^{iz}+e^{-iz}}\right)^2$

Then, from RHS,

$$1 + \tan^2 z \Rightarrow 1 + \left(\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right)^2$$

$$= 1 + \frac{(e^{iz}-e^{-iz})^2}{-(e^{iz}+e^{-iz})} = 1 + \frac{e^{2iz}-2e^{iz-iz}+e^{-2iz}}{-(e^{2iz}+2e^{iz-iz}+e^{-2iz})}$$

$$= 1 + \frac{e^{2iz}-2+e^{-2iz}}{-(e^{2iz}+2+e^{-2iz})} = \frac{\cancel{-e^{2iz}}-2-\cancel{e^{-2iz}}+\cancel{e^{2iz}}-2+\cancel{e^{-2iz}}}{-(e^{2iz}+2+e^{-2iz})}$$

$$= \frac{\cancel{-}4}{\cancel{-}(e^{2iz}+2+e^{-2iz})} = \frac{4}{e^{2iz}+2+e^{-2iz}} = \left(\frac{2}{e^{iz}+e^{-iz}}\right)^2 = \sec^2 z$$

Hence,

$$\sec^2 z = 1 + \tan^2 z$$

(b) $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

From LHS, $\sin(z_1+z_2)$ if $w = z_1+z_2$ then, $\sin w = \frac{e^{iw}-e^{-iw}}{2i}$

$$\Rightarrow \sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$$

Also, from RHS,

$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$= \left[\frac{e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}}{4i}\right] + \left[\frac{e^{i(z_1+z_2)}-e^{i(z_1-z_2)}+e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}}{4i}\right]$$

$$= \frac{e^{i(z_1+z_2)}+\cancel{e^{i(z_1-z_2)}}-\cancel{e^{i(z_2-z_1)}}-e^{-i(z_1+z_2)}+e^{i(z_1+z_2)}-\cancel{e^{i(z_1-z_2)}}+\cancel{e^{i(z_2-z_1)}}-e^{-i(z_1+z_2)}}{4i}$$

$$= \frac{2e^{i(z_1+z_2)}-2e^{-i(z_1+z_2)}}{4i} = \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}$$

Hence,

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

(1c) $\cosh(z_1 - z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2$

From LHS, $\cosh(z_1 - z_2) = \dfrac{e^{z_1 - z_2} + e^{z_2 - z_1}}{2}$

Also, from RHS,

$\cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2 = \left(\dfrac{e^{z_1} + e^{-z_1}}{2}\right)\left(\dfrac{e^{z_2} + e^{-z_2}}{2}\right) - \left(\dfrac{e^{z_1} - e^{-z_1}}{2}\right)\left(\dfrac{e^{z_2} - e^{-z_2}}{2}\right)$

$$= \frac{\cancel{e^{z_1+z_2}} + e^{z_1 - z_2} + e^{z_2 - z_1} + \cancel{e^{-(z_1+z_2)}} - \cancel{e^{z_1+z_2}} + e^{z_1 - z_2} + e^{z_2 - z_1} - \cancel{e^{-(z_1 + z_2)}}}{4}$$

$$= \frac{\cancel{2}(e^{z_1 - z_2} + e^{z_2 - z_1})}{\cancel{2} \cdot 2} = \frac{e^{z_1 - z_2} + e^{z_2 - z_1}}{2}$$

Hence, $\cosh(z_1 - z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2$ □

(1d) $\coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$

Let $w = \coth^{-1} z \Rightarrow z = \coth w = \dfrac{\cosh w}{\sinh w}$

$z = \dfrac{e^w + e^{-w}}{e^w - e^{-w}} \Rightarrow z(e^w - e^{-w}) = e^w + e^{-w}$

$\Rightarrow (z-1)e^w = (z+1)e^{-w}$

$\Rightarrow \dfrac{z+1}{z-1} = \dfrac{e^w}{e^{-w}}$

$\Rightarrow \dfrac{z+1}{z-1} = e^{2w}$

for a complete solution, $e^{2w} = e^{2(w - k\pi i)}$ for $k = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \dfrac{z+1}{z-1} = e^{2(w - k\pi i)}$

Taking ln (log) of bothsides, we have

$\ln\left(\dfrac{z+1}{z-1}\right) = 2(w - k\pi i)$ for $k = 0, \pm 1, \pm 2, \ldots$

$w - k\pi i = \dfrac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right)$ for $k = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow w = k\pi i + \dfrac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right)$ for $k = 0, \pm 1, \pm 2, \ldots$

Choosing the principal branch of $\coth z$, $k = 0$, then;

$w = \dfrac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right)$

$\Rightarrow \coth^{-1} z = \dfrac{1}{2}\ln\left(\dfrac{z+1}{z-1}\right)$ //

2ai) $\lim_{z\to 2i} \frac{(z+3)(z-7)}{2z+i} = \frac{(2i+3)(2i-7)}{2(2i)+i} = \frac{-4-14i+6i-21}{4i+i} = \frac{-8i-25}{5i}$

$= -\frac{(8i-25)}{5i} = -\frac{8}{5} + 5i = \underline{5i - \frac{8}{5}}$

ii) $\lim_{z\to 4i} \frac{z^2+25}{z-5i} = \lim_{z\to 4i} \frac{(z+5i)\cancel{(z-5i)}}{\cancel{z-5i}} = \lim_{z\to 4i} z+5i = 4i+5i = \underline{9i}$

iii) $\lim_{n\to\infty} \frac{2n-3i}{2n+i}$ Dividing through by the highest power of n, we have;

$\lim_{n\to\infty} \frac{2n-3i}{2n+i} = \lim_{n\to\infty} \frac{2-\frac{3i}{n}}{2+i/n} = \frac{2}{2} = \underline{1}$

2b.) Proof;

Suppose $f(z)$ is continous at z_0 then, for any $\varepsilon > 0 \ \exists \ |f(z)-f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$. In particular, $|f(z)-f(z_0)| < \varepsilon$ whenever $0 < |z-z_0| < \delta$

$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$.

Conversely, Suppose $\lim_{z\to z_0} f(z) = f(z_0)$ then, for any $\varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0 \ \exists \ |f(z)-f(z_0)| < \varepsilon$ whenever $0 < |z-z_0| < \delta$.

However, at $z = z_0$

$|f(z)-f(z_0)| = |f(z_0)-f(z_0)| = 0$.

for the whole interval, $|z-z_0|$, we have $|f(z)-f(z_0)| < \varepsilon$

Thus $|f(z)-f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$

$\Rightarrow f(z)$ is contn at z_0.

2c) To investigate the continuity of a complex function, $\lim_{z\to z_0} f(z) = f(z_0)$.

for $z = i \Rightarrow f(i) = 4+4i$

and $\lim_{z\to z_0} f(z) = \lim_{z\to i} f(z) = \lim_{z\to i} \left(\frac{3z^4-2z^3+8z^2-2z+5}{z-i}\right)$

$= \lim_{z\to i} \left[\frac{\cancel{(z-i)}(3z^3+3z^2i-2z^2-2zi+5z+5i)}{\cancel{z-i}}\right] = 3(i)^3+3(i)^2i-2(i)^2-2(i)i+5(i)+5i$

$= -\cancel{3i}+\cancel{3i}+2+2+5i+5i = \cancel{4+10i} \ 4i+4$

$\Rightarrow \lim_{z\to i} f(z) \neq f(i)$

Hence, $f(z)$ is ~~not~~ cont. at $z = i$

(3a) Proof:

Since $f(z)$ is contn in the region say R, then for any $\varepsilon > 0$ $\exists$ $\delta(\varepsilon) > 0$ $\ni$ $|f(z)-f(z_0)| < \varepsilon$ whenever $|z-z_0| < \delta$. which is true for any arbitrary $z_0 \in R$.

Let $z = x+iy$ and $z_0 = x_0 + iy_0$ where x and x_0 are the real part and y and y_0 are the imaginary part of z and z_0 respectively.

Then, $|f(z)-f(z_0)| = |f(x+iy) - f(x_0+iy_0)| = |f(x-x_0) + if(y-y_0)|$

$= |f(x) - f(x_0) + i(f(y) - f(y_0))| \leq |f(x) - f(x_0)| + |i||f(y) - f(y_0)| < \varepsilon$.

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ and $|f(y) - f(y_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

$\Rightarrow |x - x_0 + i(y-y_0)| \leq |x - x_0| + |i||y - y_0| < \delta$

$\Rightarrow |x - x_0| < \delta$ and $|y - y_0| < \delta$

Therefore, $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ when-ever $|x - x_0| < \delta$ and $|f(y) - f(y_0)| < \varepsilon$ whenever $|y - y_0| < \delta$.

$\Rightarrow f(x)$ is contn at x_0 which is the real part and $f(y)$ is contn at y_0 which is the imaginary part. $\square$,

(3bi) $\lim\limits_{n\to\infty} \dfrac{2n-5i}{2n+3i} = 1$

Proof:

For any $\varepsilon > 0$, $\exists$ $\delta > 0$ $\ni$ $|f(z) - 1| < \delta$ whenever $0 < |z-1| < \delta$.

$\Rightarrow \left|\dfrac{2n-5i}{2n+3i} - 1\right| < \varepsilon$ whenever $0 < |z-1| < \delta$.

$\Rightarrow \left|\dfrac{\cancel{2n} - 5i - \cancel{2n} - 3i}{2n+3i}\right| = \left|\dfrac{-8i}{2n+3i}\right| = \dfrac{8}{|2n+3i|} \leq \dfrac{8}{|2n| - |3i|} = \dfrac{8}{2n-3}$

$\leq \dfrac{8}{2n-4} = \dfrac{4}{n-2}$ so that if we choose $\varepsilon > 0$

then $\dfrac{4}{n-2} < \varepsilon$ provided $n - 2 > \dfrac{4}{\varepsilon}$ or $n > \dfrac{4}{\varepsilon} + 2$

Therefore, $|z_n - 1| = \dfrac{8}{|2n+3i|} \leq \dfrac{8}{2n-4}$ $\forall n \geq N$.

$\therefore \lim\limits_{n\to\infty} \dfrac{2n-5i}{2n+3i} = 1$

ii) To investigate the convergence of $\sum_{n=0}^{\infty} \frac{(2-i)^n}{(z-i)^{n+1}}$ we use the ratio test.

$\Rightarrow \lim_{n\to\infty} \left|\frac{Z_{n+1}}{Z_n}\right| = \lim_{n\to\infty} \left| \frac{(2-i)^{n+1}}{(z-i)^{n+2}} \div \frac{(2-i)^n}{(z-i)^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{(2-i)^{n+1}}{(z-i)^{n+2}} \cdot \frac{(z-i)^{n+1}}{(2-i)^n} \right| = \lim_{n\to\infty} \left| \frac{2-i}{z-i} \right|$.

$\Rightarrow \frac{|2-i|}{|z-i|} < 1$ provided $|z-i| > |2-i|$ or $|z-i| > \sqrt{5}$

Hence the test fail, thus, the series diverge.

3iii) A series $\sum_{n=1}^{\infty} Z_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |Z_n|$ converges while a ~~s.q~~ a series (absolute) $\sum_{n=1}^{\infty} |Z_n|$ does not converge but $\sum_{n=1}^{\infty} Z_n$ converges then the series $\sum_{n=1}^{\infty} Z_n$ is said to be conditionally convergent.

4a) Since $\sum_{n=1}^{\infty} Z_n$ converges to A and $\sum_{n=1}^{\infty} W_n$ converges to B then the sequences of their partial sums $\{S_n\}$ and $\{K_n\}$ converges to A and B respectively.
For any $\varepsilon > 0$, $\exists$ $N_1(\varepsilon), N_2(\varepsilon) \ni |S_n - A| < \varepsilon/2$ $\forall n \geq N_1$ and
$|K_n - B| < \varepsilon/2$ $\forall n \geq N_2$

$\therefore |(S_n - K_n) - (A-B)| = |S_n - A - K_n - B| \leq |S_n - A| + |-(K_n - B)| < \varepsilon/2 + \varepsilon/2$
$= \varepsilon$ $\forall n \geq N_1, N_2$

So that if we choose $N = \max(N_1, N_2)$ then
$|(S_n - K_n) - (A-B)| < \varepsilon$ $\forall n \geq N$

Since the sequence of their partial sums $S_n - K_n$ converges to $A - B$ then the series $\sum_{n=1}^{\infty} (Z_n - W_n)$ also converges to $A - B$. □

4b) Proof:
Suppose $\sum_{n=1}^{\infty} Z_n$ converges, then the sequence of partial sum $\{S_n\}$ also converges.
Suppose it converges to S i.e $S_n \rightarrow S$.
then, $S_n = Z_1 + Z_2 + \cdots + Z_n$ and S_{n-1} (Not necessary Z_0) $= Z_1 + Z_2 + \cdots + Z_{n-1}$

$\Rightarrow S_n - S_{n-1} = Z_n$

$\Rightarrow \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} Z_n$

$\Rightarrow \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} Z_n \Rightarrow S - S = \lim_{n\to\infty} Z_n$

$\Rightarrow \lim_{n\to\infty} Z_n = 0$ □

(4c i) $\lim_{n\to\infty} z^n = 0$ if $|z|<1$

Given $\varepsilon>0 \ \exists \ |z_n - z_0| < \varepsilon$ whenever $n \geq N$

$\Rightarrow |z^n - 0| = |z^n| = |z|^n < \varepsilon$ whenever $n \geq N$

So that if $\varepsilon>0$ is given, then $|z|^n < \varepsilon$ provided

$\log(|z|^n) < \log\varepsilon$

$\rightarrow n\log|z| < \log\varepsilon \Rightarrow n > \frac{\log\varepsilon}{\log|z|}$

Since $\log|z|<0$ because $|z|<1$

If we choose N to be the smallest integer greater than $\frac{\log\varepsilon}{\log|z|}$ then,

$|z^n - 0| = |z|^n < \varepsilon \ \forall n \geq N.$ □

(ii) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$

$$\Rightarrow \lim_{n\to\infty}\left[\sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right] = \lim_{n\to\infty}\left[\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right] = \lim_{n\to\infty}\left[\frac{1}{\sqrt{n+1}+\sqrt{n}}\right]$$

$$= \frac{1}{\infty} = \underline{\underline{0}}$$

(5a) Let $f(z) = U(x,y) + iV(x,y)$ and $f(z)$ is differentiable at z_0 where $z_0 = x_0 + iy_0$. If we let $\Delta z = \Delta x + i\Delta y$ then;

$\Delta U = U(x_0+\Delta x, y_0+\Delta y) - U(x_0,y_0)$ and $\Delta V = V(x_0+\Delta x, y_0+\Delta y) - V(x_0,y_0)$

If $f'(z_0) = a+ib \Rightarrow \lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = a+ib$

$$\Rightarrow \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta U + i\Delta V}{\Delta x + i\Delta y} = a+ib$$

OR

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \mathrm{Re}\left\{\frac{\Delta U + i\Delta V}{\Delta x + i\Delta y}\right\} = a \quad \text{and} \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \mathrm{Im}\left\{\frac{\Delta U + i\Delta V}{\Delta x + i\Delta y}\right\} = b$$

Suppose $\Delta y \to 0$ then,

$$\lim_{\Delta x\to 0}\frac{U(x_0+\Delta x, y_0) - U(x_0,y_0)}{\Delta x} = a \quad \text{and} \quad \lim_{\Delta x\to 0}\frac{V(x_0+\Delta x, y_0) - V(x_0,y_0)}{\Delta x} = b \quad (1)$$

$$\Rightarrow \frac{\partial U}{\partial x} = a \quad \text{and} \quad \frac{\partial V}{\partial x} = b$$

On the other hand, if $\Delta x \to 0$ then;

$$\lim_{\Delta y\to 0}\frac{V(x_0, y_0+\Delta y) - V(x_0,y_0)}{\Delta y} = a \quad \text{and} \quad \lim_{\Delta y\to 0} -\frac{[U(x_0,y_0+\Delta y) - U(x_0,y_0)]}{\Delta y} = b \quad (2)$$

$$\Rightarrow \frac{\partial V}{\partial y} = a \quad \text{and} \quad -\frac{\partial U}{\partial y} = b$$

(6b) $\oint_C \frac{ze^{2z}+1}{(z-3)(z^2-1)}\,dz$ where C: rectangle $-2 \le x \le 2;\ -\frac{3}{2} \le y \le 3/2$

$$\Rightarrow \oint_C \frac{ze^{2z}+1}{(z-3)(z^2-1)}\,dz = \int_{C_1} \frac{ze^{2z}+1}{(z-3)(z-1)}\,dz + \int_{C_2} \frac{ze^{2z}+1}{(z-3)(z+1)}\,dz$$

But $\int_{C_1} \frac{ze^{2z}+1}{(z-3)(z-1)}\,dz$ $\cancel{\text{at } z_0} = -1$

$$= 2\pi i\left[\frac{-e^{-2}+1}{8}\right] = \frac{2\pi i}{8}(1-e^{-2}) = \frac{\pi i}{4}(1-e^{-2})$$

Also,

$\int_{C_2} \frac{ze^{2z}+1}{(z-3)(z+1)}\,dz = \text{at } z = 1$

$$= 2\pi i\left[\frac{e^{2}+1}{-4}\right] = \frac{-\pi i}{2}(e^2+1)$$

$$\Rightarrow \oint \frac{ze^{2z}+1}{(z-3)(z^2-1)}\,dz = \int_{C_1} \frac{ze^{2z}+1}{(z-3)(z-1)}\,dz + \int_{C_2} \frac{ze^{2z}+1}{(z-3)(z+1)}\,dz$$

$$= \frac{\pi i}{4}(1-e^{-2}) - \frac{\pi i}{2}(e^2+1)$$

$$= \pi i\left[\frac{(1-e^{-2})}{4} - \frac{(e^2+1)}{2}\right]$$

$$= \pi i\left[\frac{1-e^{-2}-2e^2-2}{4}\right]$$

$$= \frac{-\pi i}{4}(e^{-2}+2e^2+1)$$